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BEHAVIOR OF THE VORTICITY VECTOR IN SUPERSONIC AXISYMMETRIC SWIRL FLOWS BEHIND A DETONATION WAVE

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The behavior of the vorticity vector on a discontinuity surface arising in a supersonic nonuniform combustible gas flow with the formation of a shock or detonation wave is studied. In the general case, it is a vortex flow with prescribed distributions of parameters. It is demonstrated that the ratio of the tangential component of vorticity to density remains continuous in passing through the discontinuity surface, while the quantities proper become discontinuous. Results calculated for flow vorticity behind a steady-state detonation wave in an axisymmetric supersonic flow of a combustible mixture of gases are presented.

Key words: *vorticity, shock wave, detonation wave, axisymmetric flow, discontinuity surface, conservation law.*

Formulas for components of vorticity behind a steady-state shock wave for flows with constant parameters in the general case were obtained in [1] under the assumption of an infinite shock-wave intensity. Formulas for components of vorticity behind a shock wave of an arbitrary intensity for a uniform incoming flow were derived in [2, 3]. The vorticity behind a curvilinear steady-state detonation wave in a supersonic vortex flow is obtained in the present paper.

Calculation of Vorticity behind a Steady-State Detonation Wave. Let there be a detonation wave in a steady-state supersonic axisymmetric vortex flow of a combustible gas. The detonation wave is considered as a strong discontinuity surface on which combustion of a unit mass of the gas releases an amount of heat Q, which may be a variable function. For $Q = 0$, there is a usual shock wave. In this case, the gas motion is described by the following system of equations:

$$v\,\frac{\partial \rho}{\partial r} + \rho\,\frac{\partial v}{\partial r} + \frac{\rho v}{r} + u\,\frac{\partial \rho}{\partial x} + \rho\,\frac{\partial u}{\partial x} = 0,$$

$$\rho\Big(v\,\frac{\partial u}{\partial r} + u\,\frac{\partial u}{\partial x}\Big) + \frac{\partial p}{\partial x} = 0, \qquad \rho\Big(v\,\frac{\partial v}{\partial r} + u\,\frac{\partial v}{\partial x} - \frac{w^2}{r}\Big) + \frac{\partial p}{\partial r} = 0, \tag{1}$$

$$v\,\frac{\partial w}{\partial r} + u\,\frac{\partial w}{\partial x} + \frac{vw}{r} = 0, \qquad v\,\frac{\partial}{\partial r}\Big(\frac{p}{\rho^\gamma}\Big) + u\frac{\partial}{\partial x}\Big(\frac{p}{\rho^\gamma}\Big) = 0.$$

Here $u(x,r)$, $v(x,r)$, and $w(x,r)$ are the components of velocity $\boldsymbol{V}$ in a cylindrical coordinate system (x,r,φ) and ρ and p are the density and pressure of the gas, respectively.

The vorticity $2\boldsymbol{\omega} = \operatorname{rot}\boldsymbol{V}$ has the components

$$2\omega_r = -\frac{\partial w}{\partial x}, \qquad 2\omega_x = \frac{1}{r}\,\frac{\partial rw}{\partial r}, \qquad 2\omega_\varphi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial r}. \tag{2}$$

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Relations (1) are valid in the entire domain of the gas flow; the laws of conservation of mass, momentum, and energy are satisfied on the discontinuity surface proper:

$$\rho v_n = \rho_0 v_{0n}, \qquad p + \rho v_n^2 = p_0 + \rho_0 v_{0n}^2,$$
$$\frac{1}{2} v_n^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{1}{2} v_{0n}^2 + \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} + Q, \tag{3}$$
$$v_\tau = v_{0\tau}, \qquad v_\varphi = w_0.$$

Here the quantities with the subscript 0 refer to gas parameters ahead of the wave, and the quantities without any subscript refer to gas parameters behind the wave, $v_n = \boldsymbol{V\nu}$ is the normal component of velocity, and $v_\tau = \boldsymbol{V\tau}$ and v_φ are the tangential components of velocity. The vector $\boldsymbol{\tau}$ is located in the plane of the meridional section.

The geometric conditions of compatibility are satisfied on the discontinuity surface:

$$f_{,i} = f_{,n}\nu_i + f_{,s}\tau_i, \qquad i \equiv r, x. \tag{4}$$

The comma here means the derivative with respect to the corresponding coordinate, ν_r and ν_x are the components of the unit vector of the normal to the surface, whose direction coincides with the flow direction behind the front, τ_r and τ_x are the components of the unit tangential vector to the discontinuity surface $\boldsymbol{\tau} = \boldsymbol{r}_{,s}$, and $\boldsymbol{r} = (x, r, \varphi)$. Conditions (4) relate the derivatives of a certain function f with respect to spatial coordinates to the derivative with respect to the normal $f_{,n}$ and the derivative with respect to the surface coordinate $f_{,s}$; s is a natural coordinate introduced on the discontinuity surface along the meridional section by the plane $\varphi = \text{const}$.

Using conditions (4) and the fact that the tangential components of velocity are not changed in passing through the discontinuity surface, we find expressions for the vorticity components (2) directly ahead of the wave and behind it for $r = R(x)$ [$R(x)$ is the equation of the discontinuity surface]. The vorticity directly ahead of the wave is

$$2\omega_{0x} = \frac{1}{r}\frac{\partial r w_0}{\partial r} = w_{0,n}\nu_r + w_{0,s}\tau_r + \frac{w_0}{r}, \qquad 2\omega_{0r} = -\frac{\partial w_0}{\partial x} = -w_{0,n}\nu_x - w_{0,s}\tau_x,$$
$$2\omega_{0\varphi} = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial r} = v_{0,n}\nu_x - u_{0,n}\nu_r + v_{0,s}\tau_x - u_{0,s}\tau_r; \tag{5}$$

the vorticity directly behind the wave is

$$2\omega_x = \frac{1}{r}\frac{\partial r w}{\partial r} = w_{,n}\nu_r + w_{0,s}\tau_r + \frac{w_0}{r}, \qquad 2\omega_r = -\frac{\partial w}{\partial x} = -w_{,n}\nu_x - w_{0,s}\tau_x,$$
$$2\omega_\varphi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial r} = v_{,n}\nu_x - u_{,n}\nu_r + v_{,s}\tau_x - u_{,s}\tau_r. \tag{6}$$

In these expressions, the vorticity components include the derivatives of velocity components along the normal and along the natural coordinate. The derivatives along the normal are determined from the equations of motion (1) as follows:

— ahead of the wave,

$$u_{0,n} = -\frac{1}{v_{0n}}\Big(\frac{1}{\rho_0}\frac{\partial p_0}{\partial x} + v_{0\tau}u_{0,s}\Big), \qquad v_{0,n} = -\frac{1}{v_{0n}}\Big(\frac{1}{\rho_0}\frac{\partial p_0}{\partial r} + v_{0\tau}v_{0,s} - \frac{w_0^2}{r}\Big),$$
$$w_{0,n} = -\frac{v_{0\tau}}{v_{0n}}\, w_{0,s} - \frac{w_0}{r}\Big(\nu_r + \frac{v_{0\tau}}{v_{0n}}\,\nu_x\Big); \tag{7}$$

— behind the wave,

$$u_{,n} = -\frac{1}{v_n}\Big(\frac{1}{\rho}\frac{\partial p}{\partial x} + v_{0\tau}u_{,s}\Big), \qquad v_{,n} = -\frac{1}{v_n}\Big(\frac{1}{\rho}\frac{\partial p}{\partial r} + v_{0\tau}v_{,s} - \frac{w_0^2}{r}\Big),$$
$$w_{,n} = -\frac{v_{0\tau}}{v_n}\, w_{0,s} - \frac{w_0}{r}\Big(\nu_r + \frac{v_{0\tau}}{v_n}\,\nu_x\Big). \tag{8}$$

Substituting the expressions for the derivatives along the normal (7) and (8) into Eqs. (5) and (6), we obtain the following expressions:

— for the vorticity components ahead of the wave,

$$2\omega_{0x} = (w_{0,s} + \nu_x w_0/r)(\nu_x - \nu_r v_{0\tau}/v_{0n}), \qquad 2\omega_{0r} = (w_{0,s} + \nu_x w_0/r)(\nu_r + \nu_x v_{0\tau}/v_{0n}),$$
$$2\omega_{0\varphi} = -p_{0,s}/(\rho_0 v_{0n}) + \nu_x w_0^2/(r v_{0n}) - v_{0\tau,s} v_{0\tau}/v_{0n} - v_{0n,s}; \tag{9}$$

— for the vorticity components behind the wave [with allowance for the relations on the discontinuity (3)],

$$2\omega_{x} = (w_{0,s} + \nu_x w_0/r)(\nu_x - \nu_r v_{0\tau}/v_{n}), \qquad 2\omega_{r} = (w_{0,s} + \nu_x w_0/r)(\nu_r + \nu_x v_{0\tau}/v_{n}),$$
$$2\omega_{\varphi} = -p_{,s}/(\rho v_{n}) + \nu_x w_0^2/(r v_{n}) - v_{0\tau,s} v_{0\tau}/v_{n} - v_{n,s}. \tag{10}$$

We decompose the vorticity vector into the normal ω_n and tangential components ω_τ and ω_φ:
— ahead of the wave, we have

$$2\omega_{0n} = 2(\omega_{0x}\nu_x + \omega_{0r}\nu_r) = w_{0,s} + \nu_x w_0/r,$$

$$2\omega_{0\tau} = 2(\omega_{0x}\tau_x + \omega_{0r}\tau_r) = v_{0\tau}(r w_{0,s} + \nu_x w_0)/(r v_{0n}) = 2\omega_{0n} v_{0\tau}/v_{0n},$$

$$2\omega_{0\varphi} = -p_{0,s}/(\rho_0 v_{0n}) + \nu_x w_0^2/(r v_{0n}) - v_{0\tau} v_{0\tau,s}/v_{0n} - v_{0n,s};$$

— behind the wave, we have

$$2\omega_{n} = 2(\omega_{x}\nu_x + \omega_{r}\nu_r) = w_{0,s} + \nu_x w_0/r = 2\omega_{0n},$$

$$2\omega_{\tau} = 2(\omega_{x}\tau_x + \omega_{r}\tau_r) = v_{0\tau}(r w_{0,s} + \nu_x w_0)/(r v_{n}) = 2\omega_{0\tau} v_{0n}/v_{n} = 2\omega_{0\tau}\rho/\rho_0,$$

$$2\omega_{\varphi} = -p_{,s}/(\rho v_{n}) + \nu_x w_0^2/(r v_{n}) - v_{0\tau} v_{0\tau,s}/v_{n} - v_{n,s}.$$

It follows from these relations that the normal component of vorticity remains a continuous function in passing through the discontinuity surface. The expression for the tangential component of vorticity ω_τ implies that the ratio ω_τ/ρ remains continuous in passing through the discontinuity surface, regardless whether the discontinuity is a shock wave or a detonation wave:

$$\omega_\tau/\rho = \omega_{0\tau}/\rho_0.$$

Thus, for the class of flows described, the conservation law for the quantity ω_τ/ρ is satisfied, though the quantities ω_τ and ρ become discontinuous, as in the case with unsteady flows [4].

To find the tangential component of vorticity ω_φ behind the wave, we use the laws of conservation of mass and momentum on the discontinuity surface (3) and the expression for $\omega_{0\varphi}$ in (9):

$$\nu_x w_0^2/r - v_{0\tau} v_{0\tau,s} = 2\omega_{0\varphi} v_{0n} - p_{0,s}/\rho_0 + v_{0n} v_{0n,s},$$

$$p_{,s} = p_{0,s} + p_{0,s} v_{0n}(v_{0n} - v_n) + p_0 v_{0n,s}(v_{0n} - v_n) + p_{0,s} v_{0n}(v_{0n,s} - v_{n,s}), \qquad v_n = \rho_0 v_{0n}/\rho.$$

Substituting these relations into Eq. (10) for ω_φ, we obtain

$$\omega_\varphi = \omega_{0\varphi}\frac{\rho}{\rho_0} + \frac{v_{0n,s}}{2}\Big(1 - \frac{\rho}{\rho_0}\Big)^2\Big/\frac{\rho}{\rho_0} - \frac{p_{0,s}}{2\rho_0 v_{0n}}\Big(1 - \frac{\rho}{\rho_0}\Big) - \frac{v_{0n}\rho_{0,s}}{2\rho_0}\Big(1 - \frac{\rho_0}{\rho}\Big).$$

Thus, the vorticity components behind the wave in a cylindrical coordinate system are determined as follows:

$$\omega_x = \omega_n\nu_x + \omega_\tau\tau_x = \nu_x\omega_{0n} - \nu_r\omega_{0\tau}\frac{\rho}{\rho_0}, \qquad \omega_r = \omega_n\nu_r + \omega_\tau\tau_r = \nu_r\omega_{0n} + \nu_x\omega_{0\tau}\frac{\rho}{\rho_0},$$
$$\omega_\varphi = \omega_{0\varphi}\frac{\rho}{\rho_0} + \frac{1}{2}\,(u_{0,s}\nu_x + v_{0,s}\nu_r - \varkappa v_{0\tau})\Big(1 - \frac{\rho}{\rho_0}\Big)^2\Big/\frac{\rho}{\rho_0} - \frac{p_{0,s}}{2\rho_0 v_{0n}}\Big(1 - \frac{\rho}{\rho_0}\Big) - \frac{v_{0n}\rho_{0,s}}{2\rho_0}\Big(1 - \frac{\rho_0}{\rho}\Big). \tag{11}$$

Here $\varkappa = -(\partial\boldsymbol{\nu}/\partial s)\cdot\boldsymbol{\tau}$ is the curvature of the discontinuity surface. The vorticity components behind the wave depend on the vorticity ahead of the wave, gas parameters and their derivatives along the natural coordinate, and also on the density ratio and on the functions $\varkappa$, $\boldsymbol{\nu}$, and $\boldsymbol{\tau}$ responsible for the geometry of the discontinuity surface. For constant parameters of the incoming flow, Eqs. (11) transform to the known formulas [1–3].

Vorticity behind a Steady-State Detonation Wave for Flows with Constant Parameters. If the velocity component in the direction of changing of the angular coordinate equals zero ($w_0 = 0$), and the initial parameters of the gas are constant, then the only component of vorticity (11) other than zero is

$$\omega_\varphi = -\frac{\varkappa v_{0\tau}}{2}\Big(1 - \frac{\rho}{\rho_0}\Big)^2 \Big/ \frac{\rho}{\rho_0}$$

($v_{0\tau} = u_0 \cos\alpha$ is the tangential component of velocity).

Thus, for constant parameters of the incoming flow, vorticity is a function depending on the density ratio ρ/ρ_0, wave curvature $\varkappa$, angle between the tangential line and the discontinuity surface α, and free-stream velocity u_0.

In turn, the value of ρ/ρ_0 is determined from the conservation laws on the discontinuity surface (3) and is a function of parameters determining the state of the medium and the amount of heat supplied to a unit mass of the gas $q = 2Q(\gamma^2 - 1)/a_0^2$:

$$\frac{\rho}{\rho_0} = \frac{1 + \gamma M_{0n}^2 + \sqrt{(M_{0n}^2 - 1)^2 - qM_{0n}^2}}{(\gamma - 1)M_{0n}^2 + q/(\gamma + 1) + 2}. \tag{12}$$

Here $M_{0n} = \mathrm{M}_0 \sin\alpha$, $\mathrm{M}_0 = u_0/a_0$ is the Mach number, a_0 is the velocity of sound, and γ is the ratio of specific heats.

As the radicand in Eq. (12) has to be non-negative, it follows that the flow around the body is possible for α varied in the interval $\alpha_J \leqslant \alpha \leqslant \pi/2$. Here α_J is the angle between the tangential line and the wave at the point where the wave passes to the Chapman–Jouguet regime:

$$\alpha_J = \arcsin\Big(\frac{1}{\mathrm{M}_0}\sqrt{\frac{2 + q + \sqrt{q(4 + q)}}{2}}\Big). \tag{13}$$

Relation (13) allows one to find the restrictions on problem parameters: $0 \leqslant q \leqslant q_*$. Here $q_* = (\mathrm{M}_0 - 1/\mathrm{M}_0)^2$, $(M_0)_* \leqslant \mathrm{M}_0 \leqslant \infty$, and $(M_0)_* = \sqrt{(2 + q + \sqrt{q(4 + q)})/2}$.

The tangent of the angle between the tangential line and the wave is

$$\tan\,\alpha_J = \sqrt{\frac{2 + q + \sqrt{q(4 + q)}}{2(\mathrm{M}_0^2 - 1) - q - \sqrt{q(4 + q)}}}.$$

For the shock wave, we have $\tan\,\alpha_J = \sqrt{1/(\mathrm{M}_0^2 - 1)}$. For critical values of the problem parameters $q = q_*$ or $\mathrm{M}_0 = (M_0)_*$, we obtain $\tan\,\alpha_J = \infty$ and $\alpha = \pi/2$. The angle between the tangent line and the wave decreases with increasing M_0 and increases with increasing heat release.

It was demonstrated in [5] that the transition to the Chapman–Jouguet regime in flows with a cylindrical or spherical detonation wave occurs at a finite distance, in contrast to flows with plane waves. An overdriven wave passing to the Chapman–Jouguet regime has a third-order tangency.

Let us consider the flow behind a detonation-wave segment preceding the occurrence of the Chapman–Jouguet regime, where the gas parameters are constant. As an example, we assume that a wave defined as $R^2 = 1/(a-x) - b$ is formed in a flow around a certain body. At the point x_J (point of the transition to the Chapman–Jouguet regime), the wave has the following properties (tangency of the third order): $R(x_J) = \pm\tan\,\alpha_J x_J$, $R'(x_J) = \pm\tan\,\alpha_J$, and $R''(x_J) = 0$; the derivative $R'''(x_J)$ exists. The function R equals zero at the point $x_0 = a - 1/b$ and the wave has a straight-line form behind the point $x = x_J$.

The conditions at the transition point to the Chapman–Jouguet regime are satisfied for the following values of parameters of the function: $a = 3\tan^{-2/3}\alpha_J/2$, $b = 3\tan^{2/3}\alpha_J/4$, and $x_J = \tan^{-2/3}\alpha_J/2$. The wave curvature is determined as

$$\varkappa = -\frac{\partial \boldsymbol{\nu}}{\partial s}\,\boldsymbol{\tau} = -\frac{R''}{(1 + (R')^2)^{3/2}} = -\frac{2(a - x)^2(3 - 4b(a - x))}{(1 + 4(a - x)^4R^2)^{3/2}}.$$

It is a positive function in the interval $[x_0, x_J]$. For $x \geqslant x_J$, the curvature equals zero. In this case, the wave in the entire flow region can be described by the function

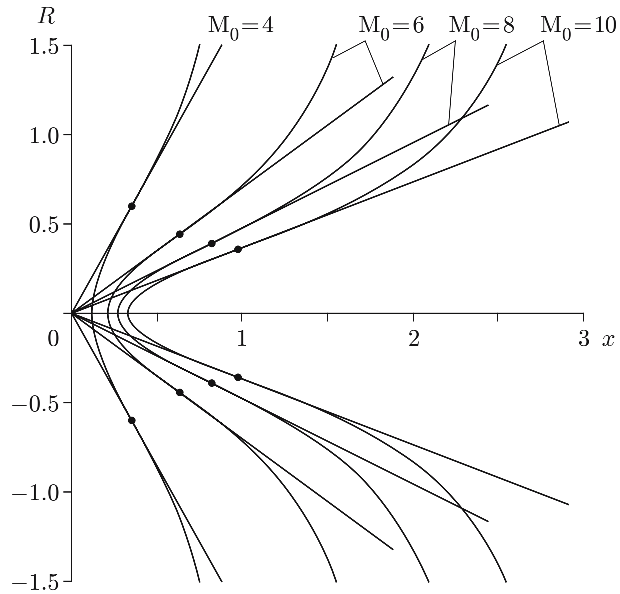

Fig. 1. Detonation wave $R(x)$ and the tangential line to the wave for $q = 10$ and $\gamma = 1.4$ (the points indicate the wave transition to the Chapman–Jouguet regime).

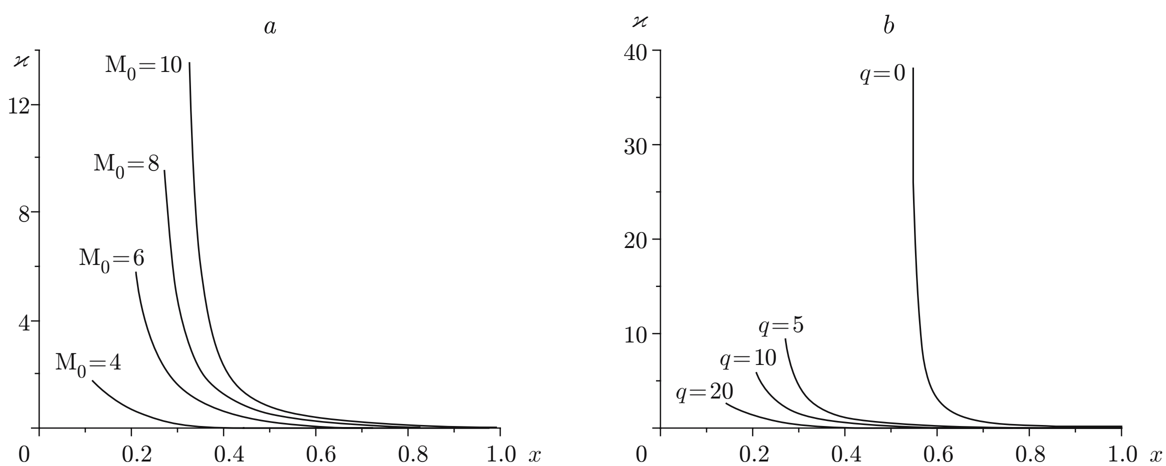

Fig. 2. Wave curvature $\varkappa$ in the interval $[x_0, x_J]$ for $\gamma = 1.4$: (a) $q = 10$; (b) $\mathrm{M}_0 = 6$.

$$R(x) = \begin{cases} \pm\dfrac{\tan^{1/3}\alpha_J}{2}\sqrt{\dfrac{6\tan^{2/3}\alpha_J x - 1}{3 - 2\tan^{2/3}\alpha_J x}}, & x_0 \leqslant x \leqslant x_J, \\ \pm\tan\alpha_J x, & x \geqslant x_J. \end{cases}$$

The detonation wave $R(x)$ and the tangential line to the wave for $q = 10$, $\gamma = 1.4$, and different values of the Mach number M_0 is illustrated in Fig. 1. The points indicate the wave transition to the Chapman–Jouguet regime. The waves behind these points are straight lines. As it follows from Fig. 1, the greater the Mach number, the smaller the angle between the tangential line and the wave and the closer the wave to the body surface.

Figures 2a and 2b show the behavior of the wave curvature $\varkappa$ in the interval $[x_0, x_J]$ for different values of the Mach number and heat release, respectively. The curvature acquires the maximum value at the point x_0 and vanishes at the transition point to the Chapman–Jouguet regime. The curvature reaches the greatest value for the shock wave ($q = 0$); the wave curvature behind the detonation wave decreases with increasing energy release in the wave.

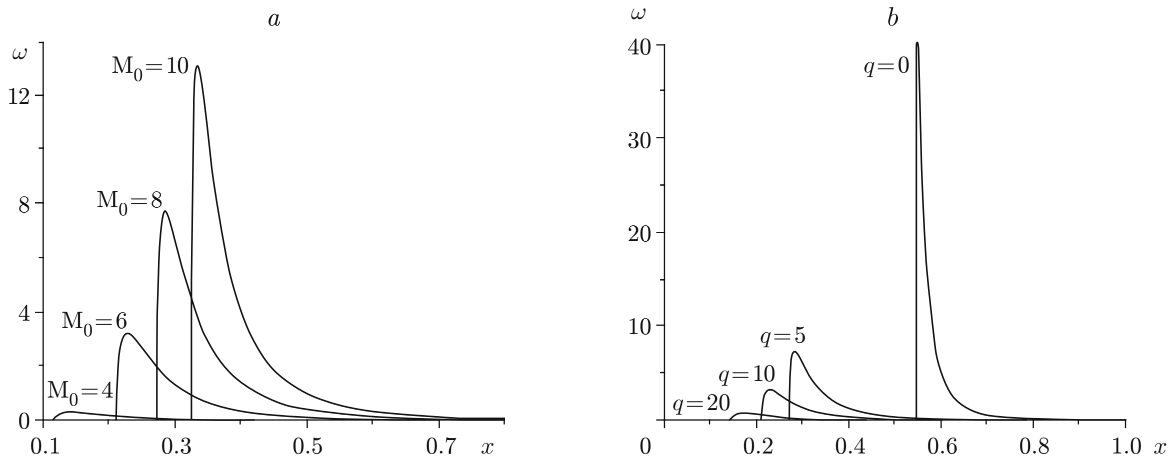

Fig. 3. Vorticity ω for $\gamma = 1.4$: (a) $q = 10$; (b) $M_0 = 6$.

Figures 3a and 3b show the vorticity $\omega = -2\omega_\varphi/u_0$ for different values of the Mach number and heat release, respectively. The vorticity equals zero at the point x_0 where the wave is normal ($\alpha = \pi/2$) and at the transition point to the Chapman–Jouguet regime x_J. The vorticity reaches the maximum value for x close to x_0. The vorticity increases with increasing Mach number (Fig, 3a) and reaches the greatest value behind the shock wave (Fig. 3b). An increase in heat release substantially reduces the vorticity behind the detonation wave.

Thus, the vorticity of a swirl flow directly behind a steady-state detonation wave is studied. The vorticity is found to reach the maximum value for shock waves and to decrease with increasing energy release in the wave for detonation waves. In passing through the discontinuity surface, the normal component of vorticity remains a continuous function. It is also demonstrated that the law of conservation of the quantity ω_τ/ρ is satisfied for the class of flows examined for all distributions of gas parameters in the incoming flow.

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